Solitons in a Grassmannian σ model coupled to a Chern-Simons term

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We study a Grassmannian σ model coupled to a Chern-Simons term. In the presence of a novel topological term the model admits exact self-dual vortex solutions which are identical to those of a pure Grassmannian model, but the topological charge has a physical meaning as a magnetic flux since the gauge field is no longer auxiliary. We also extend the theory to a noncommutative plane and analyze the BPS solutions.

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I. INTRODUCTION

The O(3) nonlinear σ model [1] and its generalization to $CP(N)$ and Grassmannian target space $\mathrm{Gr}(N,M) \equiv SU(N)/[SU(N-M)\times U(M)]$ in $2+1$ dimensions [2] have attracted a great deal of interest [3] especially due to the exact solvability. They admit the exact self-dual Bogomol'nyi-Prasad-Sommerfield (BPS) soliton solutions characterized by the second homotopy, $\pi_2(S^2) = \pi_2[\mathrm{Gr}(N,M)] = \mathbb{Z}$ [4].

The fact that the models can be described in a simple manner with the introduction of the auxiliary gauge fields or the composite gauge connections motivated people to make the gauge fields dynamical either by adding the Maxwell term or the Chern-Simons term [5,6]. The latter especially has been considered in the context of the fractional spin and statistics [7]. However, gauging the nonlinear σ model has been focused on O(3) or $CP(N-1) = \mathrm{Gr}(N,1)$ models up to now. This fact plus the recent upsurge of interest in the solitons of noncommutative field theory [8] motivate us to look at the gauged Grassmannian σ model, since noncommutativity inevitably implies the non-Abelian structure in the theory [9].

In this paper, we consider a Grassmannian σ model coupled to the non-Abelian Chern-Simons theory [10] and show that the model is still solvable in the presence of a novel topological term. The model admits exact self-dual vortex solutions which are identical to those of a pure Grassmannian model as in [2,11]. However, the topological charge has a physical meaning as a magnetic flux since the gauge field is no longer auxiliary. We also extend the theory to a noncommutative plane and analyze the BPS solutions.

The Lagrangian we propose is with the metric $\eta = \mathrm{diag}(-,+,+)$:

$$\mathcal{L} = \mathrm{tr}\Big[-(D_\mu\phi)^\dagger(D^\mu\phi) - \frac{i}{2}\tilde{\kappa}\epsilon^{\mu\nu\rho}F_{\mu\nu}(\phi^\dagger D_\rho\phi) - (\phi^\dagger\phi - I_{M\times M}M)\lambda\Big] + \mathcal{L}_{CS}, \tag{1}$$

where the matter field ϕ is an $N\times M$ complex matrix, λ is an $M\times M$ matrix valued Lagrangian multiplier, and $I_{M\times M}$ is the $M\times M$ identity matrix. $\tilde{\kappa}$ is the coupling constant of the newly introduced interaction term which is topological being independent of the metric. The "tr" is the trace over the $U(M)$ local gauge indices. We take the field ϕ to be in the anti-fundamental representation so that the $U(M)$ gauge transformation is given with $U \in U(M)$ as

$$\phi \to \phi U^\dagger, \qquad A_\mu \to U A_\mu U^\dagger + iU\partial_\mu U^\dagger. \tag{2}$$

The $U(M)$ covariant derivative is $D_\mu\phi = \partial_\mu\phi + i\phi A_\mu$, and the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. The Chern-Simons term is

$$\mathcal{L}_{\mathrm{CS}} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\,\mathrm{tr}\Big(A_\mu\partial_\nu A_\rho - \frac{2}{3}iA_\mu A_\nu A_\rho\Big). \tag{3}$$

Here κ is the Chern-Simons coefficient which should be quantized to be consistent at the quantum level. As for commutative non-Abelian cases, i.e., $M \geqslant 2$ and also for any noncommutative case [12], $2\pi\kappa$ must be an integer. In the next section, we will see that the aforementioned BPS solutions with the nonzero magnetic flux exist when $\kappa + \tilde{\kappa} = 0$. The extension to the noncommutative plane will be done in Sec. III and the final section contains some comments and discussions.

II. BPS SOLUTIONS

Before obtaining the BPS solutions, we first write the equations of motion,

$$0 = [-i(D^\mu\phi)^\dagger\phi + i\phi^\dagger D^\mu\phi] + \epsilon^{\mu\nu\rho} \times [(\tfrac{1}{2}\kappa + \tilde{\kappa})F_{\nu\rho} - i\tilde{\kappa}(D_\nu\phi)^\dagger(D_\rho\phi)], \tag{4}$$

$$0 = -D^\mu D_\mu\phi + \frac{i}{2}\tilde{\kappa}\epsilon^{\mu\nu\rho}D_\mu\phi F_{\nu\rho} + \phi\lambda, \tag{5}$$

and the Grassmannian constraint coming from the variation of the Lagrangian multiplier,

$$\phi^\dagger\phi = I_{M\times M}. \tag{6}$$

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For the Euclidean plane it is often convenient to introduce the complex coordinates,

$$z=\frac{1}{\sqrt{2}}(x_1+ix_2), \qquad \bar{z}=\frac{1}{\sqrt{2}}(x_1-ix_2), \tag{7}$$

and hence $\partial=1/\sqrt{2}(\partial_1-i\partial_2)$, $\bar{\partial}=1/\sqrt{2}(\partial_1+i\partial_2)$, $A_z=1/\sqrt{2}(A_1-iA_2)$, $A_{\bar{z}}=1/\sqrt{2}(A_1+iA_2)$.

In order to obtain the BPS equations one needs to consider the expression of the energy. The contribution to the total energy, E, comes only from the matter part,

$$\begin{aligned} E&=\int \mathrm{d}^2x\ \mathrm{tr}[(D_0\phi)^\dagger D_0\phi+(D_z\phi)^\dagger D_z\phi+(D_{\bar{z}}\phi)^\dagger D_{\bar{z}}\phi] \\ &=\begin{cases} \int \mathrm{d}^2x\ \mathrm{tr}[(D_0\phi)^\dagger D_0\phi+2(D_{\bar{z}}\phi)^\dagger D_{\bar{z}}\phi]-2\pi Q \\ \int \mathrm{d}^2x\ \mathrm{tr}[(D_0\phi)^\dagger D_0\phi+2(D_z\phi)^\dagger D_z\phi]+2\pi Q \end{cases} \\ &\geqslant|2\pi Q|, \end{aligned} \tag{8}$$

where $2\pi Q$ can be written as a boundary term using the Grassmannian constraint,

$$\begin{aligned} 2\pi Q&\equiv i\int \mathrm{d}^2x\ \mathrm{tr}[\epsilon^{ij}(D_i\phi)^\dagger D_j\phi] \\ &=\int \mathrm{d}^2x\ \mathrm{tr}[i\epsilon^{ij}D_i(\phi^\dagger D_j\phi)+F_{12}] \\ &=\oint_\infty \mathrm{d}\vec{l}\cdot \mathrm{tr}(i\phi^\dagger\vec{D}\phi+\vec{A}). \end{aligned} \tag{9}$$

The saturation of the energy occurs when the BPS or anti-BPS equations are satisfied,

$$\begin{aligned} &0=D_0\phi, \qquad 0=D_{\bar{z}}\phi: \quad \text{BPS equations,} \\ &0=D_0\phi, \qquad 0=D_z\phi: \quad \text{anti-BPS equations.} \end{aligned} \tag{10}$$

In either case these BPS or anti-BPS equations determine the gauge field completely. As $\phi^\dagger D_\mu\phi$ is anti-Hermitian with the Grassmannian constraint, the BPS or anti-BPS equations imply for all $\mu=0,1,2$:

$$0=\phi^\dagger D_\mu\phi \Leftrightarrow A_\mu=i\phi^\dagger\partial_\mu\phi. \tag{11}$$

Consequently $F_{\mu\nu}=iD_\mu\phi^\dagger D_\nu\phi-(\mu\leftrightarrow\nu)$ and $2\pi Q$ in Eq. (9) reduce simply to the magnetic flux.

For the above BPS or anti-BPS equations to be compatible with the full equations of motion (4) and (5), it is necessary to set $\tilde{\kappa}+\kappa=0$. With the BPS equations including their implication (11), the whole equations of motion reduce to

$$0=(\kappa+\tilde{\kappa})F_{12}, \tag{12}$$

and $\lambda=\pm F_{12}$ where the upper/lower sign corresponds to the BPS/anti-BPS, respectively. We emphasize here that our BPS solutions are not restricted to the conventional static limit, but can be time dependent. Thus for the BPS states with nontrivial flux to exist we should set $\tilde{\kappa}=-\kappa$, and this is the very reason why we introduced the $\tilde{\kappa}$ term. Consequently when κ is quantized as in commutative non-Abelian cases or any noncommutative case, so is $\tilde{\kappa}$. Namely $\tilde{\kappa}$ should be an integer divided by 2π at the quantum level. It is worthwhile to note that the BPS equations imply the vanishing of the electric field, $F_{0i}=0$ and hence the BPS vortices are purely magnetic.

As the anti-BPS solutions can be obtained simply by $z\leftrightarrow\bar{z}$, henceforth we focus on solving the BPS equations only. We first factorize $\phi=WH$ that is a product of an $N\times M$ matrix, W, and an $M\times M$ Hermitian matrix, H. Then the Grassmannian constraint (6) determines the Hermitian matrix, $H^2=(W^\dagger W)^{-1}$ so that

$$\phi=W\frac{1}{\sqrt{W^\dagger W}}. \tag{13}$$

Substituting this into the BPS equations yields, with $q\equiv\phi\phi^\dagger$,

$$(1-q)\partial_0W=0, \quad (1-q)\bar{\partial}W=0. \tag{14}$$

Simple solutions for these equations are provided by arbitrary time independent holomorphic matrices,

$$W=W_0(z), \tag{15}$$

on the condition that the inverse of $W_0^\dagger(\bar{z})W_0(z)$ exists.

More general solutions are available due to the peculiar $GL(M,\mathbb{C})$ "symmetry,"

$$W\rightarrow W'=W\Lambda, \tag{16}$$

where Λ is any invertible element in $GL(M,\mathbb{C})$. Under this transformation H^{-2} is covariant, q is invariant, and the solution space is preserved; if W is a solution then so is W'. We note that for $CP(N-1)$ case, i.e., $M=1$ the transformation (16) results in the ordinary gauge transformation. However for non-Abelian cases, $M\geqslant 2$ the transformation can be nontrivial. Only when $\Lambda\in U(M)$ or $\Lambda^\dagger=\Lambda^{-1}$ does the transformation reduce to the ordinary gauge transformation. For general Λ the transformation is not a symmetry of the action but generates gauge inequivalent solutions. Utilizing the "symmetry" we write the BPS solution,

$$W(t,z,\bar{z})=W_0(z)\Lambda(t,z,\bar{z}). \tag{17}$$

Now we evaluate $2\pi Q$. First, substituting Eq. (11) into the definition of Q or Eq. (9) identifies Q as the topological number [4],

$$Q=\frac{i}{2\pi}\int \mathrm{d}^2x\ \epsilon^{ij}\ \mathrm{tr}(q\partial_iq\partial_jq). \tag{18}$$

Under the local transformations $\delta q=\delta x^i\partial_iq$ we get $\delta Q=(3i/2\pi)\int \mathrm{d}^2x\ \epsilon^{ij}\ \mathrm{tr}(\delta q\partial_iq\partial_jq)=0$, which shows the topological nature of Q. For the above BPS solutions the corresponding topological number can be written as a surface integral. Straightforward calculation gives

$$Q=-\frac{1}{2\pi}\int d^2x\,\bar{\partial}\,\mathrm{tr}\left[\frac{1}{W_0^\dagger W_0}W_0^\dagger\partial W_0\right]$$
$$=\frac{i}{2\pi}\oint dz\,\mathrm{tr}\left[\frac{1}{W_0^\dagger W_0}W_0^\dagger\partial W_0\right]. \qquad (19)$$

To proceed further we write explicitly

$$W_0=(w_1,w_2,\ldots,w_M),\qquad w_a=v_a z^{k_a}+\mathcal{O}(z^{k_a-1}), \qquad (20)$$

where v_a's are N-component vectors and we can take them to be orthogonal using the $GL(M,\mathbb{C})$ "symmetry." From $\partial w_a=k_a w_a/z+\mathcal{O}(z^{k_a-2})$ it is straightforward to get the explicit value of the topological number

$$Q=-\sum_{a=1}^{M}k_a. \qquad (21)$$

For the anti-BPS solutions we obtain the positive number, $Q=\Sigma_a k_a$.

Any constant W_0 corresponds to a vacuum or a ground state. In fact, the vacuum is a pure gauge as usual. Since $W_0^\dagger W_0$ can be diagonalized by a unitary matrix, U with eigenvalues being positive, we can write

$$U^\dagger W_0^\dagger W_0 U=D:\text{diagonal},\qquad \hat{\phi}\equiv W_0 U\frac{1}{\sqrt{D}},$$
$$P(z,\bar{z})\equiv\sqrt{D}U^\dagger\Lambda(z,\bar{z}). \qquad (22)$$

Now the vacuum solution reads simply

$$\phi=\hat{\phi}\mathcal{U},\qquad A_\mu=i\mathcal{U}^\dagger\partial_\mu\mathcal{U}, \qquad (23)$$

where from Eq. (22) the constant $N\times M$ matrix $\hat{\phi}$ satisfies the Grassmannian constraint and $\mathcal{U}$ is an $M\times M$ unitary matrix, $\mathcal{U}=P(P^\dagger P)^{-1/2}$.

From the expression for the energy-momentum tensor,

$$T^{\mu\nu}=-\left|\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}\right|_{g=\eta}$$
$$=\mathrm{tr}(-D^\mu\phi^\dagger D^\nu\phi-D^\nu\phi^\dagger D^\mu\phi+\eta^{\mu\nu}D^\lambda\phi^\dagger D_\lambda\phi), \qquad (24)$$

the BPS states satisfying Eq. (10), T^{0i}, and $\epsilon_{ij}T^{0i}x^j$ vanish. Hence the BPS vortices are spinless.

III. NONCOMMUTATIVE SYSTEM

The noncommutative system is described by the same Lagrangian as the commutative one with all the multiplications replaced by the Moyal star product. The alternative and equivalent description is the operator formalism where all the fields are operators acting on a harmonic oscillator type Hilbert space, and the integration over the Euclidean plane is replaced by $2\pi\,\mathrm{Tr}$, the trace over the Hilbert space.

In the previous section, we deliberately organized every expression treating the ordering carefully so that the noncommutative generalization is to be taken straightforwardly without any reordering of the quantities. Here, we go over to the noncommutative plane and obtain the operator counterparts, BPS equations, topological number, and magnetic flux, as was done in the pure noncommutative $CP(N)$ case [13]. Our noncommutative solutions provide one example which has a well defined commutative limit.

The noncommutative plane is defined by a commutator relation, $[x_1,x_2]=i\theta$, and the Hilbert space is constructed by the induced annihilation and creation operators, $a=z/\sqrt{\theta}$, $\bar{a}=\bar{z}/\sqrt{\theta}$,

$$|n\rangle=\frac{1}{\sqrt{n!}}\bar{a}^n|0\rangle,\qquad [a,\bar{a}]=1. \qquad (25)$$

The number $\sqrt{n\theta}$ estimates the radius from the origin in the noncommutative plane so that $n\to\infty$ corresponds to the spatial infinity. The derivative of a field along the noncommutative coordinate becomes $\partial_i\phi=[\hat{\partial}_i,\phi]$ with

$$\hat{\partial}_i=\frac{i}{\theta}\epsilon_{ij}x_j. \qquad (26)$$

In particular, $\partial\phi=[-\bar{z}/\theta,\phi]$ and $\bar{\partial}\phi=[z/\theta,\phi]$.

In this way, most of the equations in the previous section can be taken freely for the noncommutative system. Here we only remark the subtle issue regarding the trace structure. Unlike the commutative case the cyclic property of the trace is not always guaranteed. In order to use the property the quantity in the trace should be localized or fall off rapidly at spatial infinity. Only in this case can one drop the trace of a commutator or the boundary term. From Eqs. (13) and (20) the derivative of the matter indeed satisfies this condition. Therefore again we can drop the boundary term in Eq. (9) and justify the cyclic property used while writing Eqs. (18) and (19).

Using $\mathrm{Tr}[a,\phi]=\lim_{n\to\infty}\langle n|\phi a|n\rangle$ we recast the contour integral expression (19) for the topological number into the form

$$Q=\mathrm{Tr}\ \mathrm{tr}([a,(W_0^\dagger W_0)^{-1}W_0^\dagger(\bar{a},W_0)])$$
$$=\lim_{n\to\infty}\langle n|\ \mathrm{tr}[(W_0^\dagger W_0)^{-1}W_0^\dagger(\hat{N},W_0)]|n\rangle, \qquad (27)$$

where $\hat{N}=\bar{a}a$ is the number operator. From $[\hat{N},z^k]=-kz^k$ and Eq. (20) we have

$$[\hat{N},w_a(z)]=-k_a w_a(z)+\mathcal{O}(z^{k_a-1}). \qquad (28)$$

This shows the topological number is the same as that of the commutative case, $Q=-\Sigma_a k_a$. Similar analysis also gives the positive number, $Q=\Sigma_a k_a$, for the anti-BPS solution.

IV. DISCUSSIONS

Thanks to the novel topological term we introduced, the Grassmannian model coupled to the Chern-Simons theory admits exact BPS solutions carrying no electric field. The topological charge is identified as the quantized magnetic flux, and this is in contrast to the ordinary O(3) nonlinear σ model coupled to the Chern-Simons term [5] where the gauge potential becomes pure gauge so that the field strength is identically zero [14]. Our solution also contrasts with the exact Jackiw-Pi vortices in the nonrelativistic Chern-Simons model [15]. They carry electric charges as well as the magnetic fluxes which are not quantized.

We point out the similarity between our BPS vortex solutions and monopoles in the $(3+1)$-dimensional Yang-Mills-Higgs model. In addition to the exact solvability both objects are electrically neutral and carry the quantized magnetic flux or charge. Furthermore they are spinless. This agrees with the close relation between the angular momentum and charges in $2+1$ dimensions. The angular momentum of a charge-flux composite is proportional to the product of the electric charge and the magnetic flux [16].

As the BPS solutions exist only when $\kappa+\tilde{\kappa}=0$, it would be interesting to see its origin in the supersymmetrized version of our model, which may dress spins to the BPS vortices as in $3+1$ dimensions [17].

Finally, we comment that the solution generating method in noncommutative theories [8] does not work in our case since it is not compatible with the Grassmannian constraint (6). The method is characterized by a nonunitary isometry, U,

$$U^{\dagger}U=I, \qquad UU^{\dagger}=(UU^{\dagger})^{2}\neq I, \tag{29}$$

where $UU^{\dagger}$ is a nontrivial projection operator on the noncommutative Hilbert space. Under the nonunitary isometry transformation as in Eq. (2), the equations of motion transform covariantly but the Grassmannian constraint is no longer satisfied.

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